Incommensurate phase on a disordered surface: Instability against the formation of overhangs and finite loops

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The stability of the quenched incommensurate phase in two dimensions against the creation of overhangs (OHs) and finite loops (FLs) in the fermion replica space is investigated for a model of domain walls with N colors. Introducing a chemical potential ϵ for OHs and FLs, the probability for the formation of these objects is studied for $\epsilon \to 0$. In the pure limit this probability vanishes with ϵ, whereas the fluctuations are long-range correlated in the quenched system. This indicates an instability related to the symmetry in replica space. It is accompanied by the creation of a massless boson. The latter leads to a power law decay with exponent $\propto 1/N$ for the product of the correlation functions along the domain walls.

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The model for the commensurate-incommensurate transition (CIT) in $d = 1+1$ dimensions is the prototype for a class of systems characterized by directed domain walls, directed random walks, directed polymers, or flux lines. The common feature is that of interacting random walks where the walkers randomly choose steps forward, left or right but not backward. The interaction is due to the condition that walks are not allowed to cross. In the presence of quenched disorder these systems should exhibit a phase which is related to a glass or frozen phase. The latter can be regarded as freezing of the domain walls (directed random walks, etc.) in a random potential. From a conceptional point of view the CIT model has attracted attention because it is soluble without disorder [1] and the effect of disorder can be regarded as a perturbation. The disordered case has been studied by various methods such as the Bethe ansatz [2], scaling arguments [3], variational approaches [4], and with perturbation theory [5]. Although these works agree on the fact that the critical exponent of the density changes from 1/2 in the pure system to 1 in the disordered system, they came to different conclusions concerning the decay of correlation functions in the quenched system. In a recent article by Tsvelik [6] it was argued that quenched disorder is irrelevant with respect to the asymptotic decay of the density-density function. This is surprising since the correlation function should be more sensitive to additional fluctuations than the density. Therefore, at least near the CIT one should expect that disorder changes the qualitative properties of the model. For the corresponding model with disorder correlated along the direction of the domain walls it was found that the long-range correlation is destroyed [7]. In the following we will present a calculation which shows that there is indeed a feature due to disorder which affects the correlations. It is characterized by the formation of overhangs (OHs) and finite loops (FLs). A crucial point in the various approaches to the CIT in the presence of disorder is the replica trick. It has been argued that there is no replica symmetry breaking (RSB) (however, see [8]). In contrast to this we will start from a broken replica symmetry given by an external field (chemical potential) which creates OHs and FLs. Using a generalized model for domain walls with N colors we see that there is RSB in the $N \to \infty$ limit. However, contributions $O(1/N)$ destroy the RSB and leave only a Kosterlitz-Thouless-like phase in replica space.

The grand canonical statistics of domain walls in $d = 1+1$ dimensions can be described by a fermion Lagrangian density in the imaginary time representation [6]

$$L = c^\dagger \partial_\tau c - \partial_x c^\dagger \partial_x c - v(x,\tau) c^\dagger c + \mu c^\dagger c \qquad (1)$$

and the partition function

$$Z = \int \mathcal{D}[c^\dagger, c] \exp\left(- \int_0^\infty d\tau \int dx L \right). \qquad (2)$$

The imaginary time τ is along the direction of the domain walls and x is perpendicular to τ. Disorder is introduced by the random potential $v(x,\tau)$, which is Gaussian distributed with zero mean, and $\langle v(x,\tau) v(x',\tau') \rangle = \sigma^2 \delta(x-x')\delta(\tau-\tau')$.

The partition function is invariant under a time-reversal transformation. Only the time differential operator is changed as $\partial_\tau \to \partial_\tau^T = -\partial_\tau$ by this transformation. The integration of the noninteracting fermions in Z gives the space-time determinant $\det(\partial_\tau + \partial_x^2 + \mu - v)$. Then the time-reversal invariance of Z is obvious because the transposition of the matrix leaves the determinant invariant.

Averaging the free energy or an expectation value can be performed either by using a supersymmetric generalization of the fermion Lagrangian [9] or by using the replica trick $\ln Z = \lim_{n\to 0}(Z^n - 1)/n$. Since most approaches to the problem presented in the literature are based on the latter, it will be used also in the following. Introducing an even number of replicas, say $2n$, we take one sector of n replicas with ∂_τ and the other sector of n replicas with ∂_τ^T. This can be written in a spinor representation for the Lagrangian density as

$$L = \begin{pmatrix} c^\dagger \\ d^\dagger \end{pmatrix} \begin{pmatrix} 0 & \partial_\tau + \partial_x^2 - v + \mu \\ \partial_\tau^T + \partial_x^2 - v + \mu & 0 \end{pmatrix} \begin{pmatrix} d \\ c \end{pmatrix}, \tag{3}$$

where c and d are n-component fermions and the matrix in (3) is diagonal with respect to the n replica components. Now we introduce a chemical potential ϵ, which allows the formation of OHs and FLs by combining the two different replica sectors. This means that the chemical potential creates a particle-hole pair $c^\dagger d$ which can be annihilated at another space-time point by $d^\dagger c$. Including this chemical potential in the Lagrangian density (3), we replace the zeros in the diagonal elements by $i\epsilon$. The imaginary unit i guarantees a positive weight for the OHs and FLs in the partition function because OHs and FLs always contain a sequence of reversed Grassmann variables which contribute a minus sign. The density of OHs and FLs can be measured by varying locally the chemical potential $\rho_\epsilon(x,\tau) \propto \langle c^\dagger(x,\tau)d(x,\tau) + d^\dagger(x,\tau)c(x,\tau)\rangle$. In the pure limit $v = 0$ this density can be evaluated by a simple calculation leading to a continuously vanishing ρ_ϵ for vanishing chemical potential ϵ. Thus the pure system is *stable* against the formation of OHs and FLs. Another interesting quantity is related to the fact that OHs and FLs are given by a combination of local edges (see Fig. 1) which are separated by a distance in x and τ. To evaluate the correlation of an edge at (x,τ) and another one at $(0,0)$ we consider

$$\langle c^\dagger(x,\tau)d(x,\tau)d^\dagger(0,0)c(0,0)\rangle. \tag{4}$$

Because this is a correlation function of non-interacting fermions it factorizes into

$$\langle c^\dagger(x,\tau)d(x,\tau)\rangle \; \langle d^\dagger(0,0)c(0,0)\rangle$$

$$-\langle c^\dagger(x,\tau)c(0,0)\rangle \; \langle d^\dagger(0,0)d(x,\tau)\rangle.$$

The first term measures the density of OHs and FLs whereas the second term is the product of correlations along the domain walls from $(0,0)$ to (x,τ). Without disorder the first term vanishes and the second term decays like $(x^2+\tau^2)^{-1}$ on large scales.

Our calculation will be based on the generalization of the domain wall model to one that has walls with N different colors. Without quenched disorder the present model separates into N independent models: individual domain walls are uniformly colored and different colors

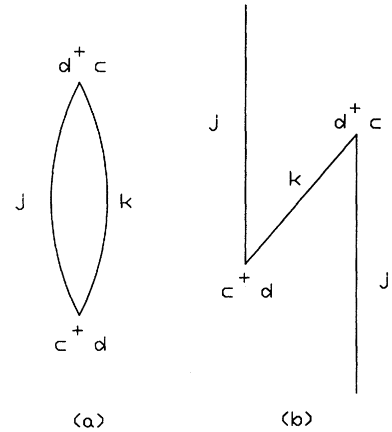

FIG. 1. (a) A finite loop and (b) an overhang created for the domain wall segments from different sectors of the replica space j and k.

do not interact. The disorder potential, on the other hand, is chosen as $v_{\alpha\beta}(x,\tau)$ with $\alpha,\beta = 1,...,N$ such that elements of the domain walls may randomly change the color from α to β. This mechanism reduces the effective hard-core repulsion because the domain walls can avoid each other by changing the color. The complex matrix elements $v_{\alpha\beta}$ are statistically independent except for the symmetry $v_{\alpha\beta} = v^*_{\beta\alpha}$.

Although the properties of the model for $N \gg 1$ might be different from those for $N = 1$, one can hope that at least the large scale properties of $N = 1$ can be obtained from the $1/N$ expansion.

After averaging we obtain an effective interaction for the fermions $[\hat{c} = (d,c)]$ in the replica model

$$\frac{\sigma^2}{2N} \sum_{i,j=1}^{2n} \sum_{\alpha,\beta=1}^{N} \hat{c}^\dagger_{i,\alpha}\hat{c}_{i,\beta}\hat{c}^\dagger_{j,\beta}\hat{c}_{j,\alpha}. \tag{5}$$

The degeneracy with respect to the N colors can be used to replace the color degree of freedom by the number of colors N. This is achieved if one decouples the effective fermion-fermion interaction of the quenched model by a different random matrix field Q, which includes fluctuations of both chemical potentials μ and ϵ: we replace $\sigma^2 N^{-1}\hat{c}^\dagger_{i,\alpha}\hat{c}_{i,\beta}\hat{c}^\dagger_{j,\beta}\hat{c}_{j,\alpha} = -\sigma^2 N^{-1}\hat{c}^\dagger_{i,\alpha}\hat{c}_{j,\alpha}\hat{c}^\dagger_{j,\beta}\hat{c}_{i,\beta}$ by $2iQ_{ji}\sum_\alpha \hat{c}^\dagger_{i,\alpha}\hat{c}_{j,\alpha}$. Q is a Hermitian $2n \times 2n$ matrix field. Going back to the replicated partition function Z^{2n} we integrate out the fermion field $\hat{c}_{i,\alpha}$. This leads to an effective action which depends only on the decoupling field as

$$S_{eff} = N\int_0^\infty d\tau \int dx \frac{2}{\sigma^2}\mathrm{Tr}_{2n}[Q(x,\tau)^2] - N \ln\det \begin{pmatrix} i\epsilon + 2iQ_{21}(x,\tau) & \partial_\tau + \partial_x^2 + \mu + 2iQ_{11}(x,\tau) \\ \partial_\tau^T + \partial_x^2 + \mu + 2iQ_{22}(x,\tau) & i\epsilon + 2iQ_{12}(x,\tau) \end{pmatrix}. \tag{6}$$

Tr_{2n} denotes the trace with respect to the $2n$ replicas and det the determinant with respect to the $2n$ replicas *and* x, τ. This is the (1+1)-dimensional replica version of the (2 + 1)-dimensional "supersymmetric" action found for the flux lines in a random potential [9].

S_{eff} depends on the number of colors only through the prefactor N. Therefore, the effective field theory for large N enables us to do the functional integration in a saddle point approximation. This approximation can be interpreted as the replacement of the chemical potentials μ and ϵ in the free fermion propagator by a self-energy term which is determined by the saddle point equation $\delta S_{eff} = 0$. There are two contributions if we assume that Q provides only additive corrections to the two chemical potentials. One is a shift of the chemical potential μ in the limit $\epsilon \to 0$

$$\mu_s \equiv 2iQ_{jj} = -\sigma^2 \int dk d\omega \frac{-k^2 + \mu + \mu_s}{D(\epsilon')} , \qquad (7)$$

with $D(\epsilon') = \epsilon'^2 + \omega^2 + (-k^2 + \mu + \mu_s)^2$, and the other is a shift of the chemical potential for OHs and FLs,

$$\epsilon' \equiv 2Q_{12} = 2Q_{21} = \epsilon' \sigma^2 \int dk d\omega / D(\epsilon'). \qquad (8)$$

$\epsilon' = 0$ is always a solution of (8). This is a replica-symmetric (RS) solution. A RSB solution $\epsilon' \neq 0$ can be found if

$$\int dk d\omega / D(0) > 1/\sigma^2. \qquad (9)$$

Since the denominator $D(\epsilon')$ is increasing with increasing ϵ'^2 there is an ϵ', that satisfies

$$\int dk d\omega / D(\epsilon') = 1/\sigma^2. \qquad (10)$$

This can be rewritten as

$$\epsilon'^{1/2} = \sigma^2 \int d\bar{k} d\bar{\omega} \frac{1}{1 + \bar{\omega}^2 + (-\bar{k}^2 + \mu'/\epsilon')^2}. \qquad (11)$$

In this case we obtain from (7) a renormalized chemical potential

$$\mu' \equiv \mu + \mu_s$$
$$= \mu - \epsilon'^{1/2} \sigma^2 \int d\bar{k} d\bar{\omega} \frac{-\bar{k}^2 + \mu'/\epsilon'}{1 + \bar{\omega}^2 + (-\bar{k}^2 + \mu'/\epsilon')^2}. \qquad (12)$$

$D(\epsilon')$ is increasing with decreasing $\mu' < 0$. This implies that there is a $\mu_c < 0$ (which depends on σ) such that $\int dk d\omega / D(0) < 1/\sigma^2$ for $\mu' < \mu_c$. Consequently, there is only a RS solution $\epsilon' = 0$ for $\mu' < \mu_c$. This behavior describes a transition from a RS solution to a RSB solution if we change μ or the disorder σ.

The evaluation of the density of domain walls in a saddle point approximation requires a cutoff because only a finite number of domain walls per unit length is reasonable. With the cutoff $k^2 = 1$ we obtain, after integration over ω,

$$\rho \propto 1 - \frac{i}{\sigma^2}(Q_{11} + Q_{22}) = 1 + \int_0^1 dk \frac{-k^2 + \mu'}{\sqrt{\epsilon'^2 + (-k^2 + \mu')^2}}. \qquad (13)$$

$\mu' < 0$ and $\epsilon' \to 0$ implies $\rho \to 0$. The density vanishes linearly. For instance, the density of the RS solution is $\rho \propto \sigma^2 + \mu$. Thus the density of the RS as well as the RSB solution is in agreement with the linear behavior of the Bethe ansatz calculation of Kardar [2].

Finally, we evaluate the Gaussian fluctuations around the saddle point solution which are the $1/N$ corrections. The stability matrix can be taken from Ref. [9]. The fluctuations of Q_{11}, Q_{22} are massive with the eigenvalues $\lambda_\pm$,

$$\lambda_+ = \lambda_- + 4 \int dk d\omega \omega^2 / D(\epsilon')^2. \qquad (14)$$

For $\epsilon' \neq 0$, λ_- is

$$\lambda_- = 4\epsilon'^2 \int dk d\omega / D(\epsilon')^2; \qquad (15)$$

i.e., the eigenvalues of the RSB solution are always positive (stable). On the other hand, for $\epsilon' = 0$ the eigenvalue λ_- is

$$\lambda_- = 2/\sigma^2 - 2 \int dk d\omega / D(0). \qquad (16)$$

Since inequality (9) holds if there are two saddle point solutions, λ_- of the RS solution (16) is negative (unstable) when the RSB solution exists. The other two eigenvalues of the RSB solution are $\lambda_3 = 2(\lambda_+ - \lambda_-) > 0$ (related to $Q_{12} - Q_{21}$) and the massless one $\lambda_4 = 0$ (related to $Q_{12} + Q_{21}$). The massless fluctuations are a consequence of a global symmetry of the model [9]. They can be expressed as a nonlinear σ model for $\mathbf{Q}$ with the constraint $\mathbf{Q}^2 = \mathbf{1}$, $\mathrm{Tr}_{2n}\mathbf{Q} = 0$, and $\mathbf{Q}\sigma_2\mathbf{Q} = -\sigma_2$. The constraint can be satisfied if we parametrize $\mathbf{Q}$ by an n-component field φ as $Q = \sigma_3(\cos^2\varphi - \sin^2\varphi) + 2\sigma_1 \cos\varphi \sin\varphi$. This parametrization neglects a unitary transformation inside each replica sector, which is not expected to affect the properties of the φ-dependent part of the model. With the field φ the action of the nonlinear σ model reads

$$S_{eff} = -\epsilon'^{-2} N \int dx d\tau \varphi(\partial_x^2 + \partial_\tau^2)\varphi. \qquad (17)$$

Formally, this is a result of a strong disorder expansion [9]. It implies that we cannot directly take the pure limit from this expression. From the bilinear action we can evaluate the correlation function of the density of edges defined in (4):

$$\langle Q_{12}(x,\tau) Q_{21}(0,0) \rangle \sim \text{const} \times \left(x^2 + \tau^2\right)^{-\epsilon'^2/2\pi N} \qquad (18)$$

for large distances. Thus there is no replica symmetry breaking for finite N because the correlation of the order parameter decays. However, disorder changes the exponent of the correlation function, which was 1 in the pure case according to the remark below (4). This correlation indicates a Kosterlitz-Thouless phase in terms of the spinor structure inside the $2n$ replicas; i.e., it is related to the correlations of the OHs and FLs. In contrast to the corresponding correlation on the pure surface, it is not only a result of the decay of the correlations along the domain walls, but comes from the combination of fluctuations of the domain walls and quenched disorder. This creates an effective correlation between different domain walls. With increasing disorder ϵ' is also increasing. Consequently, the correlation decays faster with increasing disorder, as one would expect from uncorrelated disorder. On the other hand, the correlation becomes stronger with increasing N. This can be explained by the increasing ability of domain walls to reduce "screening" due to the interaction of the disorder mediated correlation by reducing the effective hard core.

Our result is based on a saddle point calculation for an effective self-energy in the replica model with $2n$ replica and N colors. The properties of the model are similar to those of the Gross-Neveu model [10] and other (1+1)-dimensional fermion models with attractive interaction [11]. The creation of OHs and FLs in the quenched CIT model is plausible because the random potential favors OHs and FLs that can easily circumvent an unfavorable potential or freeze into the local potential wells. Since there are no OHs and FLs by definition, the replica model creates them spontaneously by combining wall elements coming from different replica sectors. However, the effect of the OHs and FLs is only marginal in our calculation because they appear with vanishing density. Nevertheless, the instability related to the OHs and FLs implies a drastic change in the quenched state compared with the pure CIT problem, as one can see from the change of the exponents.

A similar instability was also discussed in the context of the Gross-Neveu model by Witten [12], who evaluated the correlation function of the massless bosons. Those bosons also lead to a power law decay with an exponent proportional to $1/N$.

It should be emphasized that the formation of OHs and FLs is a special type of instability regarding the freezing into local minima of the random potential. Although we cannot exclude other instabilities, OHs and FLs seem to be natural candidates for a modification of the CIT due to disorder.

In conclusion, we have investigated a model for the incommensurate phase on a disordered surface, where the domain walls elements can choose randomly N colors. The $N \to \infty$ limit can be solved using a mean-field solution. Then it was found that one has (i) RSB on a mean-field level and (ii) a destruction of RSB by fluctuations analogous to the Kosterlitz-Thouless phenomenon.

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